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Theory of non-propagating surface-wave solitons

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An incompressible inviscid fluid contained in a channel in a gravitational field admits soliton-like disturbances where the velocity potential depends upon all three coordinates as well as time, yet its centre of mass can be at rest. These solitons were recently discovered by Wu, Keolian & Rudnick. The calculations are carried out with the multiple-scales approach. Consequences of mass conservation and radiation are discussed.

1. Introduction

The problem of weakly nonlinear surface waves has been considered extensively (see Ablowitz & Segur 1979; Yuen & Lake 1980; and references therein). There the approximation of phase and group velocities making a small acute angle is considered, and the stability problem as well as the permanent envelope solutions are examined.

The approximation for which phase and group velocities form a nearly 90° angle has not been fully investigated. Recent observations by Wu, Keolian & Rudnick (1984) have shown the existence of a localized stationary soliton-like disturbance in a channel of finite width b and uniform finite depth. While a finite-amplitude standing surface wave exists across the width of the channel, the amplitude of this wave is modulated along its length, suggesting then the use of this approximation.

The qualitative aspects of the problem have been already presented by Larraza & Putterman (1984) in terms of the phenomenological dispersive wave equation for sound at finite amplitude. Aranha, Yue & Mei (1982), in related work, have considered the nonlinear response of cross-waves trapped in front of a wavemaker, as an initial-boundary-value problem. They obtained numerical solutions for different inputs of the drive, and showed that the asymptotic state is governed by a nonlinear Schrödinger equation (NLS).

In this paper we consider the free oscillations of the surface displacement of a fluid of uniform finite depth d in a waveguide of constant cross-section and of infinite extent. A standing wave across the width of the channel is modulated along the length of the channel. The frequency of standing-wave motion is slightly below cutoff, with one velocity antinode. In §2 we formulate the problem, and solve it by using a multiple-scales technique. In §3 we find, up to third order in the perturbation expansion, that the equation satisfied by the modulation is a NLS equation. For $\pi d/b > 1.022$ a soliton solution is possible. We then discuss a physical interpretation of these standing-wave solitons. Section 4 deals with a discussion of mass conservation. Some final remarks are discussed in §5. Finally, we present in the Appendix an alternative derivation of equation (15).

2. Formulation of the problem

Consider the irrotational motion of an incompressible inviscid fluid in a gravitational field. The fluid at rest fills a horizontal rectangular waveguide to a depth d with $-d < z < 0$, width b with width coordinate y and of infinite extent labelled by length coordinate x. Surface-tension effects are neglected. The velocity potential satisfies Laplace's equation

$$\nabla^2\phi = 0 \quad \text{for} \quad -d < z < \zeta(x,y,t), \tag{1}$$

with boundary conditions

$$\phi_z = 0, \quad \text{at} \quad z = -d, \tag{2}$$

$$\phi_y = 0 \quad \text{at} \quad y = 0, b. \tag{3}$$

The free-surface $z = \zeta(x,y,t)$ kinematic and dynamic boundary conditions are

$$\left.\begin{aligned} \zeta_t + \zeta_x \phi_x + \zeta_y \phi_y &= \phi_z, \quad (4) \\ \phi_t + g\zeta + \tfrac{1}{2}(\nabla\phi)^2 &= 0 \quad (5) \end{aligned}\right\} \quad \text{at} \quad z = \zeta.$$

If one eliminates the surface displacement ζ in favour of the velocity potential ϕ, one obtains the following equation for ϕ valid up to terms that are cubic in derivatives of ϕ (Whitham 1976):

$$\begin{aligned} \phi_{tt} + g\phi_z = &-\left\{(\nabla\phi)_t^2 - \left[\left(\phi_z + \frac{1}{g}\phi_{tt}\right)\phi_t\right]_z\right\} \\ &-\frac{1}{2}\left\{(\nabla\phi)_x^2\phi_x + (\nabla\phi)_y^2\phi_y + (\nabla\phi)_z^2\phi_z - \frac{2}{g}[(\nabla\phi)_t^2\phi_t]_z \right. \\ &\left. -\left(\phi_z + \frac{1}{g}\phi_{tt}\right)_z\left((\nabla\phi)^2 - \frac{2}{g}(\phi_t^2)_z\right)\right\} \\ &+ O(\epsilon^4) \quad \text{at} \quad z = 0. \end{aligned} \tag{6}$$

Up to terms that are quadratic in gradients of ϕ, we find from (5)

$$g\zeta = -\phi_t + \frac{1}{2}\left\{\frac{1}{g}(\phi_t)_z^2 - (\nabla\phi)^2\right\} \quad \text{at} \quad z = 0. \tag{7}$$

In Whitham's paper there are some misprints, which we have corrected in writing (6).

We are investigating the weakly nonlinear problem of a disturbance with a high frequency of motion ω in the y-direction modulated by an envelope $\zeta_1(x,t)$ in the x-direction. Accordingly the parameter of smallness ϵ is introduced so that

$$k\zeta_{\max} = \epsilon \ll 1, \tag{8a}$$

and we seek solutions where the variations of the physical quantities are determined by

$$\frac{\mathrm{d}\log\zeta/\mathrm{d}x}{\mathrm{d}\log\zeta/\mathrm{d}y} = O(\epsilon), \quad \frac{\omega_1^2}{\omega^2} - 1 = O(\epsilon^2), \quad \frac{1}{\omega\zeta_1}\frac{\partial\zeta_1}{\partial t} = O(\epsilon^2), \tag{8b,c,d}$$

where

$$\omega_1^2 = gkT, \quad T = \tanh kd, \quad k = \frac{\pi}{b}. \tag{9}$$

Condition (8b) implies that the phase and group velocities make a nearly 90° angle.

A multiple-scales solution of Laplace's equation satisfying the boundary conditions (2) and (3) and the requirements (8) is given by

$$\phi = \left\{\phi_1(x,t)\frac{\cosh k(z+d)}{\cosh kd}\cos ky\, e^{i\omega t} + \text{c.c.}\right\}$$

$$+(\phi_0^{(2)}(x,t)\, e^{2i\omega t} + \text{c.c.}) + \frac{\cosh 2k(z+d)}{\cosh 2kd}\cos 2ky\,\{(\phi_2(x,t)\, e^{2i\omega t} + \text{c.c.}) + \phi_2^{(0)}(x,t)\}$$

$$+\left\{-\frac{1}{2k}\frac{\partial^2\phi_1}{\partial x^2}\frac{\cos ky}{\cosh kd}e^{i\omega t}\,[z\sinh k(z+d) - d\, e^{-k(z+d)}] + \text{c.c.}\right\}$$

$$+\phi_0(x+i(z+d),t) + \phi_0(x-i(z+d),t) + O(\epsilon^4), \tag{10}$$

where c.c. denotes complex conjugate.

The method of solution is to substitute (10) into (6) and equate to zero coefficients of the same $\cos(mky)\exp(ni\omega t)$ dependence.

3. Solution

To the lowest order the frequency is uniquely determined by the value of k through the dispersion relation

$$\omega_1^2 = kgT. \tag{11}$$

Thus at the linear level the system is at cutoff and the group velocity is identically zero. This result justifies our selection for the time-scale ($8d$).

To the next order, equating second harmonics with the proper y-dependence, one finds

$$(\omega^2(2k) - 4\omega^2)\,\phi_2 = \tfrac{3}{2}i\omega k^2\phi_1^2(1-T^2) \tag{12a}$$

or

$$\phi_2 = -\frac{3ik^2\phi_1^2}{8\omega T^2}(1-T^4), \tag{12b}$$

$$\phi_0^{(2)} = \frac{ik^2}{8\omega}\phi_1^2(1+3T^2), \tag{13}$$

$$\phi_2^{(0)} = 0. \tag{14}$$

where $\omega^2(2k) = 2kg\tanh 2kd$.

Equation (12) shows that the dispersion of the medium is $O(1)$. If the dispersion was $O(\epsilon)$ a resonant excitation of the second harmonic would have appeared, thus invalidating the expansion (10) (Larraza & Putterman 1984). In ($12b$) we have made use of the results of linear theory.

In a similar way, the third order gives the secular condition for the first harmonics:

$$2i\omega\frac{\partial\phi_1}{\partial t} - c^2\frac{\partial^2\phi_1}{\partial x^2} + (\omega_1^2 - \omega^2)\,\phi_1 - A\,|\phi_1|^2\,\phi_1 = 0, \tag{15}$$

where

$$c^2 = \frac{g}{2k}[T + kd(1-T^2)], \tag{16a}$$

$$A = \tfrac{1}{8}k^4(6T^4 - 5T^2 + 16 - 9T^{-2}). \tag{16b}$$

If $A > 0$, which means $kd > 1.022$, (15) possesses a soliton solution of the form

$$\phi_1 = \left(\frac{2(\omega_1^2-\omega^2)}{A}\right)^{\frac{1}{2}}\operatorname{sech}\left\{\left(\frac{\omega_1^2-\omega^2}{c^2}\right)^{\frac{1}{2}}x\right\}, \tag{17}$$

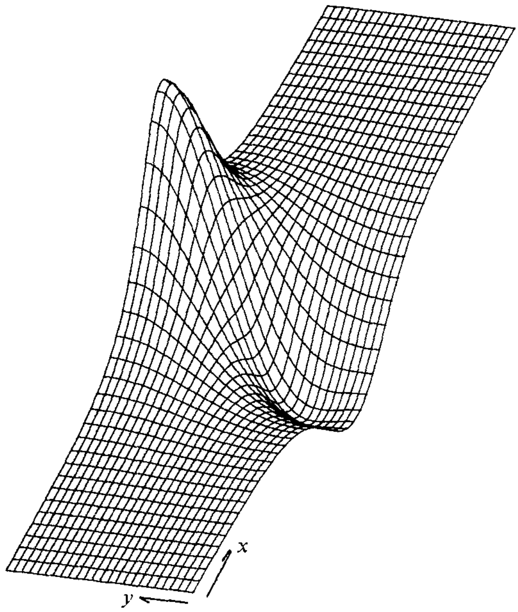

FIGURE 1. Perspective view of the solitary-wave solution (17) substituted in the equation (19) for the surface displacement.

valid in the static limit. The solution (17) has been restricted by the requirements that the frequency of motion in the small-amplitude region be given by ω. In this way we fix the phase of the standing wave across the channel. The solution (17) as it stands has the interesting feature that the amplitude and width of the modulation are determined by the depth of still water and the frequency of oscillation of the excitation. The upper limit of the latter is given by ω_1, and the lower limit is given by the depressed amplitude-dependent cutoff (for $A > 0$) or

$$\omega_1^2 - A\,|\phi_1|^2 < \omega^2 < \omega_1^2. \tag{18}$$

This amplitude-dependent cutoff is the mechanism responsible for the self-trapping of the soliton. Irrespective of the character of A, either being positive or negative, there is a depression of the height in the region where the sideband perturbation exists. To see this consider the expression for the surface elevation given by (7). Up to second order, we obtain

$$\begin{aligned} g\zeta = {} & \{-\mathrm{i}\omega\phi_1\,\mathrm{e}^{\mathrm{i}\omega t}\cos ky + \mathrm{c.c.}\} \\ & + \left\{\tfrac{1}{2}k^2\,|\phi_1|^2\,(T^2+1)\cos 2ky + \tfrac{1}{2}k^2\,|\phi_1|^2\,(T^2-1)\right. \\ & \left. -\left(\frac{k^2\phi_1^2}{4}\left(\frac{3}{T^2}-1\right)\mathrm{e}^{2\mathrm{i}\omega t}\cos 2ky + \mathrm{c.c.}\right\}\right. . \end{aligned} \tag{19}$$

The third term in (19) is the y-, t-independent shift in the surface height. According to (19), it is always negative and vanishes in the limit of infinite depth. That the mean height is less at the region where the soliton sits is not sufficient to explain its existence. Instead, we can understand this phenomenon in terms of the amplitude-dependent cutoff.

From (15) we see that the nonlinearities modify the cutoff frequency according to

$$\omega_1^2(\phi_1, d, b) = \omega_1^2(d, b) - A\,|\phi_1|^2. \tag{20}$$

For $A < 0$ the 'restoring force' will always be greater in magnitude than for the linear case, and the natural frequency of vibration is therefore increased above its linear value. For this value a soliton solution does not exist, and a modulation stability will be achieved. On the other hand, for $A > 0$ stiffness is reduced and it is possible to have a finite disturbance which in the high-amplitude region is above the effective cutoff, while in the low-amplitude region it is below cutoff. In this case the disturbance will decay exponentially in the small-amplitude region, thus trapping the energy and forming the soliton. In figure 1 we show a 3-dimensional profile of the soliton by using (19) with ϕ_1 given by (17).

4. Mass conservation

There is an apparent disagreement between our value for A and the one given by Tadjbakhsh & Keller (1960) and Miles (1976), i.e.

$$A_0 = \tfrac{1}{8}k^4(2T^4 + 3T^2 + 12 - 9T^{-2}). \tag{21}$$

Miles (1984) points out that those calculations were carried out for a closed geometry, for which local mass conservation is required. In contrast, the soliton can form at the expense of removing mass out to infinity.

Consider a situation where the channel length L is finite and there exists a localized disturbance in a region comparable to L but sufficiently far from the endwalls so that the fluid on either side is at rest owing to the quick exponential decay, which drops the disturbance to zero. The height of the fluid sufficiently far from the disturbance d_∞ now is no longer the static height d but, by mass conservation,

$$d_\infty = d - \frac{1}{2L}\int_{-L}^{L} \zeta_0\,\mathrm{d}x, \tag{22}$$

where

$$\zeta_0 = \frac{k^2|\phi_1|^2}{2g}(T^2 - 1) \tag{23}$$

is the t- and y-independent shift in height as given by the third term in (19). Thus we must interpret ω_1^2 in the NLS equation (15) as

$$\begin{aligned}\omega_1^2(d_\infty) &= gk\tanh kd_\infty \\ &= \omega_1^2(d) + \tfrac{1}{4}k^4(1-T^2)^2\frac{1}{2L}\int_{-L}^{L}|\phi_1|^2\,\mathrm{d}x.\end{aligned} \tag{24}$$

Substituting (24) into (15) yields a NLS equation with A_0 (given by (21)) instead of A (16*b*) in the limit where the static (d_∞) region is small compared with L provided that $|\phi_1|^2$ is sufficiently slowly varying.

The dynamics whereby certain disturbances are propagated to infinity is higher order ($O(\epsilon^4)$) and will be described by an equation for ϕ_0, the mean flow, which to the fourth order has the form

$$\frac{\partial^2}{\partial t^2}\{\phi_0(x+\mathrm{i}d, t) + \phi_0(x-\mathrm{i}d, t)\} + \mathrm{i}g\frac{\partial}{\partial x}\{\phi_0(x+\mathrm{i}d, t) - \phi_0(x-\mathrm{i}d, t)\} = O(\phi_1^4). \tag{25}$$

We note that for shallow water, but for kd of order unity,

$$\frac{\partial^2}{\partial t^2}\phi_0 - gd\frac{\partial^2\phi_0}{\partial x^2} = O(\phi_1^4). \tag{26}$$

5. Conclusion

Although the water-wave theory (10) requires use of three spatial coordinates, two of which yield fast variation, all of the qualitative results are the same as for the acoustic solitons (Larraza & Putterman 1984), which could be described with two spatial coordinates. The surface-wave as well as the acoustic models have Korteweg–de Vries as well as envelope solitons, which respectively have the forms

$$\zeta = \zeta(x-ut), \tag{27}$$

$$\zeta = \zeta(x-ut)\exp(\mathrm{i}\kappa x - \mathrm{i}\omega t), \tag{28}$$

where for the sound field one would replace ζ by the compression. For these solitons (27), (28) the group velocity u is given to lowest order by $\mathrm{d}\omega/\mathrm{d}\kappa$, where $\omega = \omega(\kappa)$ is the linear dispersion law for the medium in question. For the non-propagating solitons, however, the strong crosswise motion involving the y-coordinate leads to an effective decoupling of the x-component of the soliton's group velocity (Larraza & Putterman 1984). Thus there exist solutions to (15) of the form

$$\phi_1 = f(x-ut)\exp\left(-\frac{\mathrm{i}\omega u x}{c^2}\right),$$

where u can now be regarded as an additional free parameter. For KdV u is fixed by the medium and the amplitude, for the envelope soliton u is fixed by the medium and κ, and for the non-propagating solitons u can be regarded as independent of ω and π/b for a given medium. The lower bound for $|u|$ is zero. By a given medium we mean the depth of fluid in the case of surface waves.

Finally we remark that a real system has friction, and thus a drive will be required in order to maintain the localized oscillatory state. This is the only purpose of the drive. In the absence of friction a soliton once created will persist without a drive being required. Fluid motions that satisfy the above criteria are elementary excitations in the sense proposed by Landau (1947). Thus we conclude that in a fluid in an appropriate geometry at $T = 0$ K these solitons are elementary excitations of the system.

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Appendix. Expansion of the displacement

In this Appendix we calculate the nonlinear Schrödinger equation for water waves with strong crosswise motion by using the expansion,for both ζ and ϕ. The expansion for ϕ is, as before, given by (10). We remark that there are many possible expansions for ϕ consistent with (1)–(3), and the coefficient of the term in $\partial^2\phi/\partial x^2$ in (10) was uniquely determined by the requirement that

$$\lim_{h\to\infty}\frac{\partial\phi}{\partial z} < \infty.$$

The expansion for ζ, including the sum and difference frequencies that are generated by the nonlinearities, is

$$\zeta = (\zeta_1(x,t)\,e^{i\omega t} + \text{c.c.})\cos ky + \zeta_0(x,t) + [\zeta_2(x,t)\,e^{2i\omega t} + \text{c.c.} + \zeta_2^{(0)}(x,t)]\cos 2ky, \quad \text{(A 1)}$$

where ζ_0 is the t- and y-independent shift in height, which is down by a factor of ϵ with respect to ζ_1. Substituting (A 1) and (10) into (4) and (5) yields to lowest order

$$\phi_1 = \frac{-g\zeta_1}{i\omega}, \quad \omega_1 = kgT.$$

At the next order one finds for the mean and second harmonics

$$g\zeta_2^{(0)} + \tfrac{1}{2}k^2|\phi_1|^2(T^2-1) + \tfrac{1}{2}ik\,T(\phi_1\zeta_1^* - \zeta_1\phi_1^*) = 0,$$

$$\zeta_0 = \frac{k}{2T}|\zeta_1|^2(T^2-1),$$

$$2i\omega\zeta_2 - k^2\phi_1\zeta_1 = 2k(\tanh 2kh)\,\phi_2,$$

$$2i\omega\phi_2 + g\zeta_2 + \tfrac{1}{2}ik\phi_1\zeta_1 T + \tfrac{1}{4}k^2\phi_1^2(T^2-1) = 0,$$

which are clearly consistent with (12)–(14) and (19).

To the next order the terms contributing to the cubic nonlinearity of the NLS equation are

$$C(\zeta_1) = k^2\left\{\tfrac{1}{2}\zeta_2\phi_1^*(1-T^2) + \tfrac{1}{2}\phi_1\zeta_2^{(0)}(1+T^2) - \phi_1\zeta_0(1-T^2) + \left(\frac{2T^2}{1+T^2} - 2\right)\phi_2\zeta_1^*\right\}.$$

The term in ζ_0 is a measure of the mass redistribution that can occur for a localized disturbance in an infinite fluid. It accounts exactly for the difference between A and A_0. The NLS equation for ζ_1 is

$$2i\omega\frac{\partial\zeta_1}{\partial t} - c^2\frac{\partial^2\zeta_1}{\partial x^2} + (\omega_1^2 - \omega^2)\,\zeta_1 - \frac{A\omega^2}{k^2T^2}|\zeta_1|^2\zeta_1 = 0.$$

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